Homological algebra methods in the theory of Operator Algebras III, all about UCT

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27th July 2016

For applications to the C^*-algebra classification programme, it is important to compute $\mathrm{KK}(X;\cdot,\cdot)$-functor. In general, all one can hope for is some kind of spectral sequence which converges to $\mathrm{KK}(X;\cdot,\cdot)$ with the E^2-term of cohomological nature.
The most useful case is in the form of a short exact sequence of the form:

$$\mathrm{Ext}_{\mathfrak{C}}\Big(H_{*+1}(A), H_*(B)\Big) \rightarrowtail \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{C}}\Big(H_*(A), H_*(B)\Big)$$

for some homology theory H_* for C^*-algebras over X, taking values in some Abelian category $\mathfrak{C}$.

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This exact sequence should hold if A belongs to a bootstrap class adapted to our situation.

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Once we have a Universal Coefficient Theorem of this form, we can lift an isomorphism $H_*(A) \cong H_*(B)$ in $\mathfrak{C}$ to a $\mathrm{KK}(X)$-equivalence $A \simeq B$, provided A and B belong to the bootstrap class $\mathcal{B}(X)$.

Objects with projective dimension one

As usual, A has projective dimension one if there exists a $\mathfrak{I}$-projective resolution of A of the form

$$0 \longrightarrow P_1 \xrightarrow{\phi} P_0 \longrightarrow A \longrightarrow 0$$

Lemma

1. *A has projective dimension one iff it is $\mathfrak{I}^2$-projective.*
2. *Given $\mathfrak{I}$-projective resolution of A as above, there exists a $\mathfrak{I}$-equivalence $A \to \Sigma C_\phi$.*

Recall that our ideal $\mathfrak{I}$ is presupposed to be homological, i. e.

$$\mathfrak{I} = Ker\ \mathfrak{K}$$

for some stable, homological functor

$$\mathfrak{K} : \mathfrak{T} \to \mathfrak{C}$$

Theorem

Let A be an object of $\mathfrak{T}$ with the property

$$C \in Ker\mathfrak{K} \Longrightarrow \mathrm{KK}_*(X; A, C) = 0$$

and of $\mathfrak{I}$-projective dimension one. Then, for any object B of $\mathfrak{T}$, there exists a short exact sequence of the form

$$0 \to \mathrm{Ext}^1_{\mathfrak{C}}(\mathfrak{K}(A)[1], \mathfrak{K}(B)) \to \mathrm{KK}(X; A, B) \to \mathrm{Ext}^0_{\mathfrak{C}}(\mathfrak{K}(A), \mathfrak{K}(B)) \to 0.$$

Moreover

$$\mathrm{Ext}^0_{\mathfrak{T},\mathfrak{I}}(A, B) \cong \mathfrak{T}/\mathfrak{I}(A, B) \cong \mathcal{M}or_{\mathfrak{C}}(\mathfrak{K}(A), \mathfrak{K}(B))$$

and

$$\mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A, B) \cong \mathfrak{I}(A, B[1]) \cong \mathrm{Ext}^1_{\mathfrak{C}}(\mathfrak{K}(A), \mathfrak{K}(B)).$$

By assumption, A has the $\mathfrak{I}$-projective resolution dimension one, say $0 \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A$. We view it as an $\mathfrak{I}$exact chain complex of length 3. By above, there exists a commuting diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\tilde{\delta}_0} & \tilde{A} \\ \| & & \| & & \downarrow \alpha \\ P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\delta_0} & A, \end{array}$$

such that the top row is part of an $\mathfrak{I}$exact exact triangle $P_1 \xrightarrow{\phi} P_0 \to \tilde{A} \to \Sigma P_1$ and α is an $\mathfrak{I}$equivalence.

Claim: Under our assumption about A, α is an isomorphism in $\mathfrak{T}$

We embed α in an exact triangle $\Sigma B \to \tilde{A} \xrightarrow{\alpha} A \xrightarrow{\beta} B$.

$$
\begin{array}{ccccc}
 & & & & \Sigma B \\
 & & & & \downarrow \\
P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\tilde{\delta}_0} & \tilde{A} \\
\| & & \| & & \vdots \, \alpha \\
P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\delta_0} & A \\
 & & & & \downarrow \beta \\
 & & & & B
\end{array}
$$

The object B is $\mathfrak{I}$contractible because α is an $\mathfrak{I}$equivalence. Hence $\mathfrak{T}(A, B) = 0$ by our assumption on A. This forces $\beta = 0$, so that our exact triangle splits: $\tilde{A} \cong A \oplus \Sigma B$ in $\mathfrak{T}$.

First let us apply the functor $\mathfrak{T}(\cdot, B)$ to the exact triangle $P_1 \to P_0 \to \tilde{A}$. The resulting long exact sequence has form

$$\cdots \leftarrow \mathfrak{T}(P_0, B) \leftarrow \mathfrak{T}(\tilde{A}, B) \leftarrow \mathfrak{T}(\Sigma P_1, B) \leftarrow \ldots .$$

Since both P_0 and P_1 are projective, this implies that $\mathfrak{T}(\tilde{A}, B) = 0$ and hence $\mathfrak{T}(B, B) \subseteq \mathfrak{T}(\tilde{A}, B)$ vanishes as well. In particular $B \cong 0$ and α is invertible in $\mathfrak{T}$.

Now back to the proof of the theorem. Let B be arbitrary. Applying $F(\sqcup) = \mathfrak{T}(\sqcup, B)$ to the exact triangle in $\mathfrak{T}$

$$P_1 \xrightarrow{\delta} P_0 \to \tilde{A} \to \Sigma P_1$$

gives a long exact sequence

$$\cdots \leftarrow F_*(P_1) \xleftarrow{F_*(\delta)} F_*(P_0) \leftarrow F_*(A) \leftarrow F_{*-1}(P_1) \xleftarrow{F_{*-1}(\delta)} F_{*-1}(P_0) \leftarrow \cdots ,$$

We used the fact that $\tilde{A} \cong A$ in $\mathfrak{T}$. We cut this into short exact sequences of the form

$$\operatorname{coker}\big(F_{*-1}(\delta)\big) \rightarrowtail F_*(A) \twoheadrightarrow \ker\big(F_*(\delta)\big).$$

Since P_i are $\mathfrak{I}$-projective, $\mathfrak{T}(P_i, B) = \mathcal{M}or_{\mathfrak{C}}(\mathfrak{K}(P_i), \mathfrak{K}(B))$. Since moreover $\mathfrak{K}(P_i)$ are projective in $\mathfrak{C}$, $\ker F_*(\delta) = \operatorname{Ext}^0_{\mathfrak{C}}(\mathfrak{K}(A), \mathfrak{K}(B))$ and $\operatorname{coker} F_{*-1}(\delta) = \operatorname{Ext}^1_{\mathfrak{C}}(\mathfrak{K}(A)[1], \mathfrak{K}(B))$. The claimed result follows.

It is useful to recall that there are natural monomorphisms:

$$\frac{\mathfrak{T}(A,B)}{\mathfrak{I}(A,B)} \rightarrowtail \mathrm{Ext}^0_{\mathfrak{T},\mathfrak{I}}(A,B), \qquad \frac{\mathfrak{I}(A,B)}{\mathfrak{I}^2(A,B)} \rightarrowtail \mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,\Sigma B).$$

Thus $\mathfrak{I}^2(A,B)$ is the kernel of a natural map

$$\mathfrak{I}(A,B) \rightarrowtail \mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,\Sigma B).$$

The composition

$$\mathfrak{I}(A,B) \rightarrowtail \mathrm{Ext}^1_{\mathfrak{T},\mathfrak{I}}(A,B) \cong \mathrm{Ext}^1_{\mathfrak{C}}(\mathfrak{K}(A),\mathfrak{K}(B))$$

is given explicitly as follows. Given $h \in \mathfrak{I}(A,B)$, embed h in an exact triangle $\Sigma B \to C \to A \to B$. This triangle is $\mathfrak{I}$exact because h is an $\mathfrak{I}$phantom map, so that

$$\mathfrak{K}(\Sigma B) \rightarrowtail \mathfrak{K}(C) \twoheadrightarrow \mathfrak{K}(A)$$

is an exact triangle in $\mathfrak{C}$. Our map sends h to the class in

$$\mathrm{Ext}^1_{\mathfrak{C}}(\mathfrak{K}(A),\mathfrak{K}(\Sigma B))$$

determined by this extension in $\mathfrak{C}$.

Example

1. $\mathfrak{T} = C^*(pt)$ - the category of separable C*-algebras
2. $\mathcal{B}$ be the bootstrap class from yesterday
3. $F = K_*$, the K-theory functor and
4. $\mathfrak{I} = Ker K_*$.

The range category of K_* is the category of ($\mathbb{Z}/2\mathbb{Z}$-graded) abelian groups, hence has projective dimension one.
The above produces the usual UCT exact sequence for C*-algebras in $\mathcal{B}$. Note that, moreover, $\mathfrak{I}^2 = 0$ (again since every object has $\mathfrak{I}$-projective dimension one.

$\mathfrak{T} = \mathfrak{KK}(X)$

Our homology theory:

Definition

The filtrated K-theory over X comprises the collection of functors which to a C^*-algebra A over X associates $\mathbb{Z}/2$-graded Abelian groups $K_*(A(Y))$ for all locally closed subsets $Y \subseteq X$ together with all natural transformations between these functors.

The starting point for our study of filtrated K-theory is the fact that the covariant functors $A \mapsto \mathrm{K}_*(A(Y))$ are representable, that is,

Theorem

For any C^*-algebras A over X and $Y \subset X$ locally closed

$$\mathrm{K}_*(A(Y)) = \mathrm{KK}_*(X; \mathcal{R}_Y, A)$$

for suitable C^*-algebra $\mathcal{R}_Y$ over X.

Definition

Let $(X, \preceq)$ be a partially ordered set. Its *order complex* is the simplicial set $\mathrm{Ch}(X)$ whose n-simplices are *chains* $x_0 \preceq x_1 \preceq \cdots \preceq x_n$ in X and whose face and degeneracy maps delete or double an entry of the chain. We denote its simplicial realisation by $\mathrm{Ch}(X)$ as well.

Equivalently, $\mathrm{Ch}(X)$ is the classifying space of the thin category that has object set X and a morphism $x \to y$ whenever $x \preceq y$. The order complex is the main ingredient in the construction of the representing objects $\mathcal{R}_Y$ for $Y \in \mathbb{LC}(X)$.

The *non-degenerate n*-simplices in $\mathrm{Ch}(X)$ are the *strict* chains $x_0 \prec \cdots \prec x_n$ in X. We let S_X be the set of all *strict* chains. For each $I = (x_0 \prec \cdots \prec x_n) \in S_X$, we let Δ_I be a copy of Δ_n; more formally, $\Delta_I = \{(t, I) \mid t \in \Delta_n\}$. We also let $\Delta_I^\circ \subseteq \Delta_I$ be the corresponding open simplex $\Delta_n \setminus \partial\Delta_n$.
The space $\mathrm{Ch}(X)$ is obtained from the union $\coprod_{I \in S_X} \Delta_I$ by identifying Δ_I with the corresponding face in Δ_J whenever $I, J \in S_X$ satisfy $I \subseteq J$. Thus the underlying set of $\mathrm{Ch}(X)$ is a *disjoint* union

$$\mathrm{Ch}(X) = \coprod_{I \in S_X} \Delta_I^\circ. \tag{2.1}$$

For $I \in S_X$, let $\min I$ and $\max I$ be the (unique) minimal and maximal elements in S_X, respectively. We define two functions

$$m, M \colon \mathrm{Ch}(X) \to X$$

by mapping points in Δ_I° to $\min I$ and $\max I$, respectively. This well-defines functions on $\mathrm{Ch}(X)$ because of (2.1).

Lemma

If $Y \subseteq X$ is closed, then $m^{-1}(Y) \subseteq \mathrm{Ch}(X)$ is an open subset and $M^{-1}(Y) \subseteq \mathrm{Ch}(X)$ is closed. If $Y \subseteq X$ is open, then $m^{-1}(Y) \subseteq \mathrm{Ch}(X)$ is closed and $M^{-1}(Y) \subseteq \mathrm{Ch}(X)$ is open. If $Y \subseteq X$ is locally closed, then $m^{-1}(Y) \subseteq \mathrm{Ch}(X)$ and $M^{-1}(Y) \subseteq \mathrm{Ch}(X)$ are locally closed.

Let X^{op} be X with the topology for the reversed partial order $\succ$; that is, the open subsets of X^{op} are the closed subsets of X, and vice versa. We may rephrase Lemma 6 as follows:

[redacted]

The map $(m, M)\colon \mathrm{Ch}(X) \to X^{\mathrm{op}} \times X$ is continuous.

Let

$$\mathcal{R} := \mathcal{C}(\mathsf{Ch}(X))$$

be the C^*-algebra of continuous functions on $\mathsf{Ch}(X)$. Since

$$\operatorname{Prim}\mathcal{R} = \operatorname{Prim}\mathcal{C}(\mathsf{Ch}(X)) \cong \mathsf{Ch}(X),$$

the map (m, M) turns $\mathcal{R}$ into a C^*-algebra over $X^{\mathrm{op}} \times X$. We abbreviate

$$S(Y, Z) := m^{-1}(Y) \times M^{-1}(Z) \subseteq \mathsf{Ch}(X);$$

this is a locally closed subset of $\mathsf{Ch}(X)$ by Lemma 6

Definition

We define the C^*-algebra $\mathcal{R}_Y$ over X in such a way that

$$\mathcal{R}_Y(Z) = \mathcal{R}(Y^{\mathrm{op}} \times Z) = \mathcal{C}_0(S(Y, Z))$$

for all $Y, Z \in \mathbb{LC}(X)$; here Y^{op} denotes Y with the subspace topology from X^{op}. Equivalently, we let $\mathcal{R}_Y$ be the restriction of $\mathcal{R}$ to $Y^{\mathrm{op}} \times X$, viewed as a C^*-algebra over X via the coordinate projection $Y^{\mathrm{op}} \times X \to X$.

The morphism

$$\mathsf{KK}_*(X;\mathcal{R}_Y,A) \to \mathsf{K}_*(A(Y))$$

is given by

$$\phi \to \phi_*(\xi)$$

where ξ is the class of the trivial bundle on $\mathcal{R}_Y(Y) = Ch(Y)$.

Target category of filtered K-theory

Definition

$\mathcal{NT}$ is the $\mathbb{Z}/2$-graded pre-additive category given by natural transformations of filtered K-theory.

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In everyday language, using our representability theorem,

$$\mathcal{O}bj(\mathcal{NT}) = \text{ locally closed subsets of } X$$
$$\mathcal{NT}(Z, Y) = \mathrm{KK}_*(X; \mathcal{R}_Y, \mathcal{R}_Z) = K_*(\mathcal{R}_Z(Y)).$$

Target category of filtered K-theory

Definition

i) A (countable) module over $\mathcal{NT}$ is an additive functor from $\mathcal{NT}$ to the category of (countable) $\mathbb{Z}/2$-graded Abelian groups.
ii) $\mathfrak{C}$ denotes the abelian category of countable $\mathcal{NT}$-modules.

Target category of filtered K-theory

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FK is the stable homological functor from the Kasparov category $\mathfrak{KK}(X)$ of C^*-algebras over X to $\mathfrak{C}$ given by

$$A \mapsto \left\{ \begin{array}{ccc} Y & \to & \mathrm{K}_*(A(Y)) \\ \phi & \to & \mathrm{KK}(X;\phi,A) \end{array} \right\}$$

To get acquainted with this approach to natural transformations, we compute some important examples. Let $Y \in \mathbb{LC}(X)$, let $U \in \mathbb{O}(Y)$, and $Z := Y \setminus U$. There is an extension

$$\mathcal{R}_{Y\setminus U} \rightarrowtail \mathcal{R}_Y \twoheadrightarrow \mathcal{R}_U \tag{2.2}$$

of C^*-algebras over X. This means that there are C^*-algebra extensions

$$\mathcal{R}_{Y\setminus U}(Z) \rightarrowtail \mathcal{R}_Y(Z) \twoheadrightarrow \mathcal{R}_U(Z)$$

for all $Z \in \mathbb{LC}(X)$. This follows because $\mathcal{R}$ is a C^*-algebra over $X^{\mathrm{op}} \times X$. The extension (2.2) is semi-split in $\mathfrak{C}^*\mathfrak{alg}(X)$ and hence has a class in $\mathrm{KK}_1(X; \mathcal{R}_U, \mathcal{R}_Z)$ and produces an exact triangle

$$\Sigma\mathcal{R}_U \to \mathcal{R}_Z \to \mathcal{R}_Y \to \mathcal{R}_U \tag{2.3}$$

in $\mathfrak{KK}(X)$. The following lemma identifies the natural transformations corresponding to these maps between representing objects.

Lemma

The maps in the extension triangle (2.3) *correspond to the natural transformations* $\mathrm{FK}_U[1] \Leftarrow \mathrm{FK}_Z \Leftarrow \mathrm{FK}_Y \Leftarrow \mathrm{FK}_U$ *.*

Example

To make our constructions more concrete, we now consider the example $n = 2$, which corresponds to extensions of C^*-algebras. There are only three non-empty locally closed subsets: $1 = [1,1]$, $12 = [1,2]$, and $2 = [2,2]$. The order complex is an interval; we label its end points 1 and 2. The map (m, M) from $\mathrm{Ch}(X) = [1,2]$ to $X^{\mathrm{op}} \times X$ maps

$$1 \mapsto (1,1), \qquad 2 \mapsto (2,2), \qquad]1,2[\mapsto (1,2).$$

Correspondingly, we have

$$\begin{aligned} S(1,1) &= \{1\}, & S(1,2) &=]1,2[, & S(1,12) &= [1,2[,\\ S(2,1) &= \emptyset, & S(2,2) &= \{2\}, & S(2,12) &= \{2\},\\ S(12,1) &= \{1\}, & S(12,2) &=]1,2], & S(12,12) &= [1,2]. \end{aligned}$$

Taking K-theory, we get

$$\begin{aligned} \mathcal{NT}(1,1) &= \mathbb{Z}[0], & \mathcal{NT}(1,2) &= \mathbb{Z}[1], & \mathcal{NT}(1,12) &= 0,\\ \mathcal{NT}(2,1) &= 0, & \mathcal{NT}(2,2) &= \mathbb{Z}[0], & \mathcal{NT}(2,12) &= \mathbb{Z}[0],\\ \mathcal{NT}(12,1) &= \mathbb{Z}[0], & \mathcal{NT}(12,2) &= 0, & \mathcal{NT}(12,12) &= \mathbb{Z}[0]. \end{aligned}$$

Exact modules

Definition

An $\mathcal{NT}$-module M is called exact if the sequences

$$\begin{array}{ccccc} M_0(U) & \xrightarrow{i_U^V} & M_0(V) & \xrightarrow{r_V^Y} & M_0(Y) \\ \uparrow{\scriptstyle \delta_Y^U} & & & & \downarrow{\scriptstyle \delta_Y^U} \\ M_1(Y) & \xleftarrow[r_V^Y]{} & M_1(V) & \xleftarrow[i_U^V]{} & M_1(U) \end{array}$$

are exact for all $V \in \mathbb{LC}(X)^*$, $U \in \mathbb{O}(V)$, $Y := V \setminus U$. Notice that we allow U and Y to be disconnected here.

Lemma

The class of exact modules is closed under direct sums and has the two-out-of-three property for module extensions.
It contains all free modules and the filtrated K-theory of any separable C^*-algebra.

Main results

we assume that X is linearly ordered

Theorem

The following are equivalent for an $\mathcal{NT}$-module M:

- *M is a direct sum of free modules.*
- *M is projective.*
- *M is free as an Abelian group and exact.*

Theorem

The following are equivalent for an $\mathcal{NT}$-module M:

- *M has a projective resolution of length 1.*
- *M has a projective resolution of finite length.*
- *M is exact.*
- *M is in the range of filtrated K-theory.*

Hence there are $\mathcal{NT}$-modules without a projective resolution of finite length, but these cannot arise as filtrated K-groups.

Application to classification

Theorem

Filtrated K*-theory is a complete invariant for strongly purely infinite, stable, nuclear, separable C^*-algebras with primitive ideal space X and simple subquotients in the bootstrap class:*

- *Two such are isomorphic if and only if their filtrated* K*-theories are isomorphic $\mathcal{NT}$-modules.*
- *An $\mathcal{NT}$-module is the filtrated* K*-theory of such a C^*-algebra if and only if it is exact.*

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For $X = \{1, 2\}$, that is, C^*-algebra extensions, we recover a classification result of Mikael Rørdam. Our proof generalises the method of Alexander Bonkat.

A counterexample

A space for which filtrated K-theory is not enough

Topologise $Z_n := \{0, \ldots, n\}$ such that $Y \subseteq Z_n$ is open if and only if $1 \in Y$ or $Y = \emptyset$.

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- For $n \leq 2$, filtrated K-theory is a complete invariant for C^*-algebras over Z_n.
- This fails for $n = 3$.
  But we can get a complete invariant by adding another K-theory functor to filtrated K-theory.

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A space for which filtrated K-theory is not enough

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- For $n \leq 2$, filtrated K-theory is a complete invariant for C^*-algebras over Z_n.
- This fails for $n = 3$.
  But we can get a complete invariant by adding another K-theory functor to filtrated K-theory.
- It is unclear how to proceed for general n.

The ring of operations

- Since the partial order on Z_n has length 1, it is easy to describe representing objects for $\mathrm{K}_*(A(Y))$ for $Y \in \mathbb{LC}(X)^*$ and compute the relevant K-groups.

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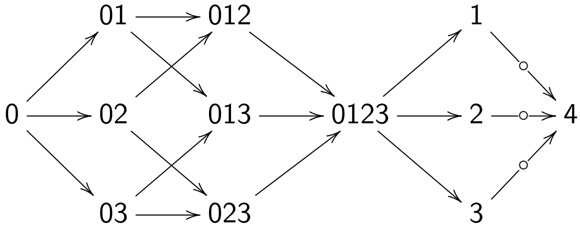
The ring of operations

- Since the partial order on Z_n has length 1, it is easy to describe representing objects for $\mathrm{K}_*(A(Y))$ for $Y \in \mathbb{LC}(X)^*$ and compute the relevant K-groups.
- They are of the form $\mathrm{K}^*(S_{YZ})$ for subsets of the star with n ends.
- The ring of natural transformations on filtrated K-theory is generated by inclusions of open subsets, restriction to closed subsets, and boundary maps.

For $n = 3$, we get the following diagram:



The relations can also be described explicitly.

What works

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- For all spaces Z_n, the ring of natural transformations has the expected generators and relations.
- It is a semi-split extension of the semi-simple ring $\mathbb{Z}^{\mathbb{LC}(Z_n)^*}$ by a nilpotent ideal.
- All projective modules are direct sums of free modules, and a module M is projective if and only if it is free as an Abelian group and $\mathrm{Tor}_1^{\mathcal{NT}}(\mathcal{NT}_{ss}, M) = 0$.

What fails

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- Exactness of a module is not sufficient to ensure that $K \mapsto K_{ss}$ preserves injective maps.

Theorem

For the space Z_3, there is an $\mathcal{NT}$-module that is exact and free as an Abelian group but not projective.
If P_Y denotes the free module $FK(\mathcal{R}_Y)$, then we may take the cokernel P of the canonical map

$$P_{0123} \to P_{012} \oplus P_{013} \oplus P_{023}.$$

This map is injective but induces the zero map on the semi-simple parts.

Counterexample to classification

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An example is the cokernel of the injective morphism

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An example is the cokernel of the injective morphism

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Theorem

There are two strongly purely infinite, stable, nuclear, separable C^*-algebras with primitive ideal space Z_3 and simple subquotients in the bootstrap class with isomorphic filtrated K-theory which are not KK^{Z_3}-equivalent.

Enriched filtered K-theory

Let $\mathcal{R}^e$ be the mapping cone of

$$\mathcal{R}_{012} \oplus \mathcal{R}_{023} \oplus \mathcal{R}_{013} \to \mathcal{R}_{0123}.$$

Let $\mathsf{FK}^e = \mathsf{FK} \cup K^e$, where

$$K^e_*(A) = \mathsf{KK}_*(Z_3; \mathcal{R}^e, A)$$

Let $\mathcal{NT}^e$ be the category of natural transformations of FK^e. The resulting enriched filtered K-theory again satisfy the main theorems, the exact modules are of projective dimension one and every exact module is in the range of enriched filtered K-theory.